The Importance of Anisotropy for Relativistic Fluids with Spherical Symmetry

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Abstract It is shown that an effective anisotropic spherically symmetric fluid model with heat flow can absorb the addition to a perfect fluid of pressure anisotropy, heat flow, bulk and shear viscosity, electric field and null fluid. In most cases the induction of effective heat flow can be avoided.

1 Introduction

Spherically symmetric perfect fluid solutions in general relativity have been studied from its very beginning, starting with the interior Schwarzschild solution. Gradually different mechanisms in stellar models have been identified that create pressure anisotropy [1, 2]. Such are the exotic phase transitions during gravitational collapse, the existence of a solid core, the presence of a type P superfluid. Viscosity may also be a source of local anisotropy as well as the slow rotation of a fluid. Radiating spherical collapse demands energy dissipation, which is usually described in two limiting cases [3, 4]. The diffusion approximation applies whenever the mean free path of photons and neutrinos inside the star is much smaller than its size. This is accounted by the introduction of heat flow. The second limit is free streaming and applies in the opposite case. This leads to the introduction of a null fluid (pure radiation). The combination of these two cases covers a large number of physical situations. More realistic fluids possess also bulk and shear viscosity [5–7]. Charged perfect fluids or dust have been discussed by many authors [8]. It has been shown too that the sum of two perfect fluids, two null fluids or a perfect and a null fluid can be represented by effective anisotropic fluid models [9]. In view of the many existing relations among the fluid models mentioned above it is hard to point out some centre.

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In this paper it is shown that the anisotropic fluid with heat flow is in some sense the most fundamental model and can absorb the addition of viscosity, charge and null fluids. Hence, it seems reasonable to study this embracing model and apply the results to the others too. For example, charged fluids are studied with no reference to anisotropy, but they are particular cases of neutral anisotropic fluids. Charged fluids with anisotropy effectively reduce to neutral anisotropic ones. The diffusion and the free streaming approximations are mathematically equivalent, but the first is easier to be investigated.

In Sect. 2 the basic anisotropic fluid model is defined. In Sect. 3 the fluid is supplied with bulk and shear viscosity which leads to a new effective anisotropic model. In Sect. 4 the same is done for the addition of charge and in Sect. 5 null fluids are accommodated into the anisotropic model. Section 6 summarizes all additions and the effective characteristics of the fundamental anisotropic model are given. Several conclusions are drawn. In Sect. 7 the static case is discussed and a formula for the general solution is presented. Section 8 contains some basic conclusions.

2 Anisotropic Fluid Model with Heat Flow

Einstein's field equations are given by

$$8\pi T_{\alpha\beta} = G_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, $T_{\alpha\beta}$ is the energy-momentum tensor (EMT) and units are used so that $c = G = 1$. The general spherically symmetric metric is written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \tag{2}$$

where A, B, R are positive functions of t and r only. The spherical coordinates are numbered as $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi$. The Einstein tensor involves the Ricci tensor and scalar which are given by the metric and its first and second derivatives [4]. Its non-trivial components are $G_{00}, G_{01}, G_{11}, G_{22} = \sin^{-2}\theta \, G_{33}$.

We are interested in the structure of EMT. It reads for anisotropic fluids with heat flow

$$T_{\alpha\beta} = (\mu + p_t) u_\alpha u_\beta + p_t g_{\alpha\beta} + (p_r - p_t) \chi_\alpha \chi_\beta + q_\alpha u_\beta + u_\alpha q_\beta. \tag{3}$$

Here μ is the energy density, p_r is the radial pressure, p_t is the tangential pressure, u^α is the four velocity of the fluid (a timelike vector), χ^α is a unit spacelike vector along the radial direction and q^α is the heat flux (in the radial direction too). We have

$$u^\alpha u_\alpha = -1, \qquad \chi^\alpha \chi_\alpha = 1, \qquad u^\alpha \chi_\alpha = 0, \qquad u^\alpha q_\alpha = 0. \tag{4}$$

It is assumed that the coordinates are comoving, hence, the fluid is motionless in them

$$u^\alpha = A^{-1}\delta_0^\alpha, \qquad \chi^\alpha = B^{-1}\delta_1^\alpha, \qquad q^\alpha = qB^{-1}\delta_1^\alpha \tag{5}$$

where $q = q(r, t)$. This gives

$$T_{00} = \mu A^2, \qquad T_{01} = -qAB, \qquad T_{11} = p_r B^2, \qquad T_{22} = p_t R^2, \tag{6}$$

which should be plugged in the Einstein equations. They become

$$8\pi\mu = A^{-2}\left(2\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R} - B^{-2}\left[2\frac{R''}{R}+\left(\frac{R'}{R}\right)^2 - 2\frac{B'R'}{BR}-\frac{B^2}{R^2}\right], \qquad (7)$$

$$8\pi p_r = -A^{-2}\left[2\frac{\ddot{R}}{R}-\left(2\frac{\dot{A}}{A}-\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + B^{-2}\left(2\frac{A'}{A}+\frac{R'}{R}\right)\frac{R'}{R}-\frac{1}{R^2}, \qquad (8)$$

$$8\pi p_t = -A^{-2}\left[\frac{\ddot{B}}{B}+\frac{\ddot{R}}{R}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)+\frac{\dot{B}\dot{R}}{BR}\right]$$
$$+ B^{-2}\left[\frac{A''}{A}+\frac{R''}{R}-\frac{A'B'}{AB}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{R'}{R}\right], \qquad (9)$$

$$4\pi q AB = \frac{\dot{R}'}{R}-\frac{\dot{B}R'}{BR}-\frac{A'\dot{R}}{AR}. \qquad (10)$$

Here the dot means time derivative and the prime is a radial derivative.

The anisotropic fluid does not radiate when $q = 0$ and becomes perfect when $p_r = p_t$. Thus it accommodates anisotropy of pressure and heat flow when one starts with a perfect fluid model.

Let us see now what happens when other EMTs are added to the basic anisotropic one.

3 Bulk and Shear Viscosity

Bulk viscosity [5–7] adds to the basic EMT the following piece

$$T_{\alpha\beta}^B = -\zeta\Theta h_{\alpha\beta} \qquad (11)$$

where ζ is a coefficient, Θ is the expansion of the fluid and $h_{\alpha\beta}$ is the projector on the hyperplane orthogonal to u^α

$$\Theta = u_{;\alpha}^\alpha, \qquad h_{\alpha\beta} = g_{\alpha\beta}+u_\alpha u_\beta. \qquad (12)$$

Obviously, this means the appearance of effective pressures

$$p_r^B = p_t^B = -\zeta\Theta, \qquad (13)$$

which should be added to p_r, p_t. They do not change the degree of anisotropy $\Delta p = p_r - p_t$. Thus even perfect fluid can absorb the bulk viscosity. The quantities μ, q remain the same. No heat flow is generated in particular.

Shear viscosity is responsible for the piece

$$T_{\alpha\beta}^S = -2\eta h_{\alpha\gamma}h_{\beta\delta}\sigma^{\gamma\delta} \qquad (14)$$

where $\sigma_{\alpha\beta}$ is the shearing tensor

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)}+a_{(\alpha}u_{\beta)}-\frac{1}{3}\Theta h_{\alpha\beta}, \qquad (15)$$

η is some coefficient and a_α is the acceleration

$$a_\alpha = u_{\alpha;\beta}u^\beta. \qquad (16)$$

The shearing tensor satisfies the conditions

$$\sigma_{\alpha\beta}u^{\beta} = 0, \qquad \sigma_{\alpha\beta}g^{\alpha\beta} = 0, \tag{17}$$

hence, (14) transforms into

$$T^{S}_{\alpha\beta} = -2\eta\sigma_{\alpha\beta}. \tag{18}$$

Use of (5), (15) gives the non-zero components of the shear [4]

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \qquad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma \tag{19}$$

where

$$\frac{2}{3}\sigma^2 = \sigma^{\alpha\beta}\sigma_{\alpha\beta}. \tag{20}$$

One can check that the same components follow when $\sigma_{\alpha\beta}$ is written as the tensor

$$\sigma_{\alpha\beta} = -\frac{1}{3}\sigma h_{\alpha\beta} + \sigma\chi_{\alpha}\chi_{\beta}. \tag{21}$$

Thus it coincides with the general shear tensor defined by (15) in the spherically symmetric case. It also satisfies relations (17) in any metric.

Plugging (21) into (18) and comparing it to (3) we find the effectively generated pressures

$$p^{S}_{r} = -2p^{S}_{t} = -\frac{4}{3}\eta\sigma. \tag{22}$$

The degree of anisotropy is changed. There is no generation of energy density or heat flow. The scalars Θ, σ can be expressed through the metric and its first derivatives:

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \qquad \sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right). \tag{23}$$

4 Electromagnetic Fields

The EMT of electromagnetic fields is given by

$$T^{EM}_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\mu}_{\beta} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right) \tag{24}$$

where $F_{\mu\nu}$ is the Faraday tensor. One defines a unit timelike vector field n^{μ}. An observer moving in its direction will measure electric and magnetic field respectively

$$E_{\alpha} = F_{\alpha\mu}n^{\mu}, \qquad H_{\alpha} = \frac{1}{2}\varepsilon_{\alpha\mu\nu}F^{\mu\nu}. \tag{25}$$

These fields are spacelike, $E^{\alpha}n_{\alpha} = H^{\alpha}n_{\alpha} = 0$. The Faraday tensor decomposes like [10]

$$F_{\alpha\beta} = \varepsilon_{\alpha\beta\mu}H^{\mu} - 2E_{[\alpha}n_{\beta]}. \tag{26}$$

Plugging this expression into (24) gives formula (7) from [10], which becomes after some rearrangements

$$T_{\alpha\beta}^{EM} = \frac{1}{4\pi}\left(H^2 + E^2\right)\left(n_\alpha n_\beta + \frac{1}{2}g_{\alpha\beta}\right) - \frac{1}{4\pi}\left(E_\alpha E_\beta + H_\alpha H_\beta\right) + 2j_{(\alpha}n_{\beta)} \qquad (27)$$

where $E^2 = E_\mu E^\mu$, $H^2 = H_\mu H^\mu$ and j_α is the Poynting vector that measures the energy flow in the spacetime

$$j_\alpha = \frac{1}{4\pi}\varepsilon_{\alpha\mu\nu}E^\mu H^\nu. \qquad (28)$$

In order to absorb this EMT by the EMT for anisotropic fluid we choose the direction $n^\alpha = u^\alpha$ and $\chi^\alpha = E^\alpha/E$. The latter is possible because when spherical symmetry is imposed $H_\alpha = 0$ and E_α has only a radial spatial component. The would be heat flow term in (27) disappears and we get

$$T_{\alpha\beta}^{E} = 2eu_\alpha u_\beta + eg_{\alpha\beta} - 2e\chi_\alpha\chi_\beta, \quad e = \frac{E^2}{8\pi}. \qquad (29)$$

Thus the addition of electric field induces effective pressures and energy density

$$\mu^E = p_t^E = -p_r^E = e, \qquad (30)$$

related by simple linear equations of state. Hence, a charged perfect fluid or a charged anisotropic fluid may be represented effectively by some neutral anisotropic fluid. There is no heat flow induction in this case.

Charged fluids, however, must satisfy the Maxwell equations in addition to the Einstein ones. It is clear from the above that the electromagnetic tensor has a single component. The first pair of Maxwell equations gives $F_{10} = \Phi'$, where Φ is the only component of the electromagnetic potential. The second pair of equations reads

$$4\pi\tau u^\alpha = F^{\alpha\beta}{}_{;\beta} \equiv (-g)^{-1/2}\left[(-g)^{1/2}F^{\alpha\beta}\right]_{,\beta} \qquad (31)$$

where τ is the charge density, g is the determinant of the metric, usual derivative is denoted by comma and the covariant derivative is denoted by semi-colon. In the spherically symmetric case this formula provides two equations. One of them gives

$$\frac{R^2}{AB}\Phi' = P(r), \qquad (32)$$

$P(r)$ being an arbitrary function of the radius. Hence, the combination in the l.h.s. depends only on the radial coordinate. Then the second equation yields

$$4\pi\tau = \frac{P'}{BR^2} \qquad (33)$$

and becomes a formula for the charge density. Plugged in the definition of E the Maxwell equations lead to

$$E = \frac{P(r)'}{R^2} \qquad (34)$$

which represents a constraint on the form of the effective density and pressures e in the general time-dependent case and no constraint in the static case.

5 Null Fluid

Null fluid describes dissipation in the free streaming approximation [4] and adds to the basic EMT the piece

$$T_{\alpha\beta}^N = \varepsilon l_\alpha l_\beta \tag{35}$$

where l^α is the null vector

$$l^\alpha = A^{-1}\delta_0^\alpha + B^{-1}\delta_1^\alpha = u^\alpha + \chi^\alpha, \tag{36}$$

satisfying the relations

$$l^\mu l_\mu = 0, \qquad l^\mu u_\mu = -1. \tag{37}$$

Substituting (36) into (35) one finds

$$T_{\alpha\beta}^N = \varepsilon u_\alpha u_\beta + \varepsilon \chi_\alpha \chi_\beta + \varepsilon \left(u_\alpha \chi_\beta + u_\beta \chi_\alpha \right). \tag{38}$$

A comparison of this expression with (3), taking into account that $q^\alpha = q\chi^\alpha$, shows that the addition of null fluid generates effective energy density, radial pressure and heat flow, all of them equal

$$\mu^N = p_r^N = q^N = \varepsilon. \tag{39}$$

No tangential pressure is generated. This is the only case where an effective heat flow is induced.

6 Summary

The results in the previous sections show that viscosity, electric charge and null fluids are equivalent to induced energy density and pressures, related by simple linear equations of state $\mu = np_r$, $n = 0, \pm 1$ and $p_t = kp_r$, $k = 0, \pm 1, -1/2$. When all such additions are combined and absorbed by the initial anisotropic fluid model, one obtains an effective model with

$$\mu^e = \mu + e + \varepsilon, \tag{40}$$

$$p_r^e = p_r - \zeta\Theta - \frac{4}{3}\eta\sigma - e + \varepsilon, \tag{41}$$

$$p_t^e = p_t - \zeta\Theta + \frac{2}{3}\eta\sigma + e + \varepsilon, \tag{42}$$

$$q^e = q + \varepsilon. \tag{43}$$

These should be plugged into (6) and hence in the l.h.s. of the Einstein equations (7)–(10). There are 4 equations for 8 functions ($\mu, p_r, p_t, q, \zeta, \eta, e, \varepsilon$). The quantities σ, Θ are given by (23). One has to impose 4 relations on these functions or set some of them to zero in order to obtain a determined system of equations. Several conclusions can be drawn.

Only the functions giving the two modes of dissipation of energy (q, ε) have effect upon the heat flow.

Viscosity (ζ, η) does not induce effective energy density.

An important characteristic of the fluid model is the anisotropy factor

$$\triangle p^e = p_r^e - p_t^e = \triangle p - 2\eta\sigma - 2e. \tag{44}$$

We see that shear viscosity and charge induce pressure anisotropy. Their absorption by a perfect fluid ($\triangle p = 0$) makes the latter anisotropic and adds several more sources of anisotropy to the usual ones [1].

Bulk viscosity and null fluid don't induce anisotropy and may be absorbed by an effective perfect fluid model with heat flow.

Charged perfect fluids are equivalent to anisotropic neutral fluids because

$$\triangle p = -2e. \tag{45}$$

In addition to the Einstein equations charged fluids satisfy Maxwell equations but we have clarified that they give a formula for the charge density and a mild constraint on e. Thus results about anisotropic fluids may be carried over to charged perfect fluids and vice versa.

7 The Static Case

In this case there is no time dependence and $R = r$, $q = 0$. The other three Einstein equations reduce to

$$8\pi\mu = \frac{1}{r^2} - \frac{B^{-2}}{r}\left(\frac{1}{r} - 2\frac{B'}{B}\right), \tag{46}$$

$$8\pi p_r = -\frac{1}{r^2} + \frac{B^{-2}}{r}\left(\frac{1}{r} + 2\frac{A'}{A}\right), \tag{47}$$

$$8\pi p_t = B^{-2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right]. \tag{48}$$

The difference of the last two equations gives a linear equation for B^{-2}. Introducing the variable z

$$A^2 = e^{2\int(z-2/r)dr} \tag{49}$$

and solving the linear equation one obtains [11]

$$B^2 = -\frac{z^2 K^2}{r^6[2\int \frac{z}{r^8}(1 + 8\pi\triangle p r^2)K^2 dr + C]} \tag{50}$$

where C is a constant of integration and

$$K = e^{\int\left(\frac{2}{zr^2} + z\right)dr}. \tag{51}$$

Thus the two generating functions $\triangle p$ and z determine the metric of any anisotropic fluid solution. Equations (46)–(48) then determine its energy density and the two pressures. In particular, using the results in the previous sections, we can plug formulas (40)–(42) into the left hand sides of (46)–(48) to obtain the general solution of any fluid with anisotropy, shear and bulk viscosity and electric charge.

8 Conclusions

Anisotropic fluid models were studied for the first time by Lemaitre in 1933 on Einstein's recommendation (see the Golden oldie [12] and the references therein). His work, however, had no impact for a long time and perfect fluid models have been examined for decades. In later times physical reasons were given for anisotropy of pressures in relativistic star models [1, 3] and now there exists an extensive literature on this topic. In the present paper we have given additional arguments in favour of the anisotropic fluid model with heat flow as an effective model, encompassing fluids with anisotropy, viscosity, charge and radiation. We have also given a vocabulary of the characteristics of different fluid models in terms of the basic anisotropic one. Hence, one can concentrate on finding its solutions and then translating them to the other models. This procedure should yield numerous, albeit formal, relations between the solutions of anisotropic and charged perfect models, or between viscous and anisotropic models and so on. Such direction of research deserves a lot of further study.

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